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ON LIE DERIVATIONS OF LIE IDEALS OF PRIME ALGEBRAS

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ABSTRACT

Let $\mathcal{F}$ be a commutative ring with 1, let $\mathcal{A}$ be a prime $\mathcal{F}$-algebra with Martindale extended centroid $\mathcal{C}$ and with central closure $\mathcal{A}_c$ and let $\mathcal{R}$ be a noncentral Lie ideal of the algebra $\mathcal{A}$ generating $\mathcal{A}$. Further, let $\mathcal{Z}(\mathcal{R})$ be the center of $\mathcal{R}$, let $\overline{\mathcal{R}} = \mathcal{R}/\mathcal{Z}(\mathcal{R})$ be the factor Lie algebra and let $\delta\colon \overline{\mathcal{R}} \to \overline{\mathcal{R}}$ be a Lie derivation. Suppose that $\mathrm{char}(\mathcal{A}) \neq 2$ and $\mathcal{A}$ does not satisfy St_{14}, the standard identity of degree 14. We show that $\mathcal{R} \cap \mathcal{C} = \mathcal{Z}(\mathcal{R})$ and there exists a derivation of algebras $D\colon \mathcal{A} \to \mathcal{A}_c$ such that $x^D + \mathcal{C} = (x+\mathcal{C})^\delta \in (\mathcal{R}+\mathcal{C})/\mathcal{C} = \overline{\mathcal{R}}$ for all $x \in \mathcal{R}$. Our result solves an old problem of Herstein.

1. Introduction

In what follows $\mathcal{F}$ is a commutative ring with 1. Let $\mathcal{A}$ be an $\mathcal{F}$-algebra and $x, y \in \mathcal{A}$. We set $[x,y] = xy - yx$ and $x \circ y = xy + yx$. It is well-known that $\mathcal{A}$ is a Lie $\mathcal{F}$-algebra with respect to **the Lie product** $[\,,\,]$. An $\mathcal{F}$-submodule $\mathcal{T}$ of $\mathcal{A}$ is called **a Lie subalgebra** of $\mathcal{A}$ if $[\mathcal{T},\mathcal{T}] \subseteq \mathcal{T}$. Next, an $\mathcal{F}$-submodule $\mathcal{T}$ of $\mathcal{A}$ is said to be **a Lie ideal** of $\mathcal{A}$ if $[\mathcal{T},\mathcal{A}] \subseteq \mathcal{T}$. Let $\mathcal{T}$ be a Lie subalgebra of $\mathcal{A}$. An

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$\mathcal{F}$-linear map d: $\mathcal{T} \to \mathcal{A}$ is said to be **a Lie derivation** if $[x, y]^d = [x^d, y] + [x, y^d]$ for all $x, y \in \mathcal{T}$. Given a nonempty subset $\mathcal{R}$ of $\mathcal{A}$, we set

$$\mathcal{Z}(\mathcal{R}) = \{x \in \mathcal{R}: [x, \mathcal{R}] = 0\}.$$

At his 1961 AMS Hour Talk, titled "Lie and Jordan Structure in Simple, Associative Rings," Herstein [17] posed a number of problems on Lie (Jordan) isomorphisms and derivations. We consider the following two of them:

PROBLEM 1: *Describe Lie derivations of prime rings* [17, Problem 3].

PROBLEM 2: *Given a prime ring $\mathcal{A}$, describe Lie derivations of $[\mathcal{A}, \mathcal{A}]$ and $[\mathcal{A}, \mathcal{A}]/\mathcal{Z}([\mathcal{A}, \mathcal{A}])$* [17, Problem 4].

The study of Lie and Jordan mappings goes back to Ancochea [1, 2], Kaplansky [25], Hua [19, 20], Jacobson and Rickart [23], Herstein [16] and Smiley [39]. Since then for almost 30 years the study of Lie isomorphisms and Lie derivations was carried on mainly by Martindale and his students [18, 21, 29, 30, 31, 32, 33, 34, 35, 38] (see also [25, 28]).

In 1964 Martindale, generalizing an unpublished result of Kaplansky (obtained in the case of a matrix ring over a field), described Lie derivations of primitive rings of characteristic not 2 with nontrivial idempotents [30]. In the subsequent papers the analogous problem was considered either in the context of prime rings with involution [21, 26, 40, 41], or in the context of von Neumann algebras [37] under a similar assumption.

In 1993 Brešar [12] solved Problem 1 under the assumption that the prime ring in question does not satisfy St_4, the standard identity of degree 4. It was the first time that functional identities were applied to obtain the description of Lie isomorphisms and Lie derivations. Since then the method of functional identities has been further developed (see [6] for a historical account) and has been successfully applied to such problems in [3, 4, 5, 8, 9, 11, 13, 14, 15, 40, 41].

Our goal is a solution of Problem 2. The present paper is based on recent results on functional identities [6, 7] briefly discussed in the second section and the following theorems are its main results.

THEOREM 1.1: *Let $\mathcal{F}$ be a commutative ring with 1, let $\mathcal{A}$ be a prime $\mathcal{F}$-algebra of characteristic not 2 with extended centroid $\mathcal{C}$, let $\mathcal{R}$ be a noncentral Lie ideal of the $\mathcal{F}$-algebra $\mathcal{A}$, let $\mathcal{B}$ be the $\mathcal{F}$-subalgebra of $\mathcal{A}$ generated by $\mathcal{R}$ and let δ: $\mathcal{R} \to \mathcal{R}$ be a Lie derivation of the Lie $\mathcal{F}$-algebra $\mathcal{R}$. Suppose that $\mathcal{A}$ does not satisfy St_{14}, the standard identity of degree 14. Then there exist a derivation $D : \mathcal{B} \to \mathcal{B}\mathcal{C} + \mathcal{C}$ of $\mathcal{F}$-algebras and an $\mathcal{F}$-linear map ζ: $\mathcal{R} \to \mathcal{C}$ such that*

(a) $\zeta([\mathcal{R},\mathcal{R}]) = 0$;
(b) $x^D = x^\delta + \zeta(x)$ *for all* $x \in \mathcal{R}$;
(c) *if* $[\mathcal{R},\mathcal{R}] = \mathcal{R}$, *then* $\mathcal{B}^D \subseteq \mathcal{B}$.

THEOREM 1.2: *Let* $\mathcal{F}$ *be a commutative ring with* 1, *let* $\mathcal{A}$ *be a prime* $\mathcal{F}$*-algebra of characteristic not* 2 *with extended centroid* $\mathcal{C}$, *let* $\mathcal{R}$ *be a noncentral Lie ideal of the* $\mathcal{F}$*-algebra* $\mathcal{A}$ *and let* $\mathcal{B}$ *be the* $\mathcal{F}$*-subalgebra of* $\mathcal{A}$ *generated by* $\mathcal{R}$. *Further, let* $\overline{\mathcal{R}} = \mathcal{R}/\mathcal{Z}(\mathcal{R})$ *be the factor Lie algebra of* $\mathcal{R}$ *by the Lie ideal* $\mathcal{Z}(\mathcal{R})$ *and let* $\delta\colon \overline{\mathcal{R}} \to \overline{\mathcal{R}}$ *be a Lie derivation of the Lie* $\mathcal{F}$*-algebra* $\overline{\mathcal{R}}$. *Suppose that* $\mathcal{A}$ *does not satisfy* St_{14}, *the standard identity of degree* 14. *Then* $\mathcal{Z}(\mathcal{R}) = \mathcal{R} \cap \mathcal{C}$ *and there exists a derivation* $D\colon \mathcal{B} \to \mathcal{BC} + \mathcal{C}$ *of* $\mathcal{F}$*-algebras such that*
(a) $(x + \mathcal{C})^\delta = x^D + \mathcal{C} \in (\mathcal{R} + \mathcal{C})/\mathcal{C} = \overline{\mathcal{R}}$ *for all* $x \in \mathcal{R}$;
(b) *if* $\mathcal{R} = [\mathcal{R},\mathcal{R}]$, *then* $\mathcal{B}^D \subseteq \mathcal{B}$.

Theorems 1.1 and 1.2 give a solution of Problem 2.

Given a (Lie) ring $\mathcal{L}$ and an element $a \in \mathcal{L}$, we denote by ad_a the map $\mathrm{ad}_a\colon \mathcal{L} \to \mathcal{L}$ defined by the rule $x^{\mathrm{ad}_a} = ax - xa$, $x \in \mathcal{L}$. Clearly ad_a is a derivation of $\mathcal{L}$. The derivation ad_a is called **the inner derivation of $\mathcal{L}$ determined by** a. Following Jacobson [22] a Lie algebra is called **complete** if its center is equal to 0 and all its derivations are inner. Further, a Lie algebra is said to be **simple complete** if it is complete and has no proper nonzero ideals which are complete Lie algebras.

THEOREM 1.3: *Let* $\mathcal{F}$ *be a field of characteristic not* 2, *let* $\mathcal{A}$ *be a prime* $\mathcal{F}$*-algebra, let* $\mathcal{R}$ *be a noncentral Lie ideal of the* $\mathcal{F}$*-algebra* $\mathcal{A}$ *with* $\mathcal{R} = [\mathcal{R},\mathcal{R}]$, *let* $\overline{\mathcal{R}} = \mathcal{R}/\mathcal{Z}(\mathcal{R})$ *be the factor Lie algebra of* $\mathcal{R}$ *by the Lie ideal* $\mathcal{Z}(\mathcal{R})$ *and let* $\mathcal{B}$ *be the* $\mathcal{F}$*-subalgebra of* $\mathcal{A}$ *generated by* $\mathcal{R}$. *Suppose that* $\mathcal{A}$ *does not satisfy* St_{14}, *the standard identity of degree* 14, *and let* $\mathrm{Der}_{\mathcal{F}}(\mathcal{B})$, $\mathrm{Der}_{\mathcal{F}}(\mathcal{R})$ *and* $\mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}})$ *be the Lie algebras of derivations of the associative algebra* $\mathcal{B}$ *and Lie algebras* $\mathcal{R}$ *and* $\overline{\mathcal{R}}$ *respectively. Then they are complete Lie algebras canonically isomorphic under the maps* $\alpha\colon \mathrm{Der}_{\mathcal{F}}(\mathcal{B}) \to \mathrm{Der}_{\mathcal{F}}(\mathcal{R})$ *and* $\beta\colon \mathrm{Der}_{\mathcal{F}}(\mathcal{R}) \to \mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}})$ *given by*

$$x^{d^\alpha} = x^d \quad \text{for all } x \in \mathcal{R}, d \in \mathrm{Der}_{\mathcal{F}}(\mathcal{B}),$$
$$(x + \mathcal{Z}(\mathcal{R}))^{d^\beta} = x^d + \mathcal{Z}(\mathcal{R}) \quad \text{for all } x \in \mathcal{R}, d \in \mathrm{Der}_{\mathcal{F}}(\mathcal{R}).$$

Further, let $\mathrm{Inn}_{\mathcal{R}}(\mathcal{B})$ *be the ideal of* $\mathrm{Der}_{\mathcal{F}}(\mathcal{B})$ *of inner derivations determined by the elements of* $\mathcal{R}$ *and let* $d \in \mathrm{Der}_{\mathcal{F}}(\mathcal{B})$. *Then* $[d, \mathrm{Inn}_{\mathcal{R}}(\mathcal{B})] = 0$ *implies that* $d = 0$.

THEOREM 1.4: *Let $\mathcal{F}$ be a field of characteristic not 2, let $\mathcal{A}$ be a simple $\mathcal{F}$-algebra, let $\mathcal{R} = [\mathcal{A}, \mathcal{A}]$ and let $\overline{\mathcal{R}} = \mathcal{R}/\mathcal{Z}(\mathcal{R})$ be the factor Lie algebra of $\mathcal{R}$ by the Lie ideal $\mathcal{Z}(\mathcal{R})$. Suppose that $\mathcal{A}$ does not satisfy St_{14}, the standard identity of degree 14, and let $\mathrm{Der}_{\mathcal{F}}(\mathcal{A})$, $\mathrm{Der}_{\mathcal{F}}(\mathcal{R})$ and $\mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}})$ be the Lie algebras of derivations of the associative algebra $\mathcal{A}$ and Lie algebras $\mathcal{R}$ and $\overline{\mathcal{R}}$ respectively. Then they are simple complete Lie algebras canonically isomorphic under the maps α: $\mathrm{Der}_{\mathcal{F}}(\mathcal{A}) \to \mathrm{Der}_{\mathcal{F}}(\mathcal{R})$ and β: $\mathrm{Der}_{\mathcal{F}}(\mathcal{R}) \to \mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}})$ given by*

$$x^{d^\alpha} = x^d \quad \text{for all } x \in \mathcal{R}, d \in \mathrm{Der}_{\mathcal{F}}(\mathcal{A}),$$
$$(x + \mathcal{Z}(\mathcal{R}))^{d^\beta} = x^d + \mathcal{Z}(\mathcal{R}) \quad \text{for all } x \in \mathcal{R}, d \in \mathrm{Der}_{\mathcal{F}}(\mathcal{R}).$$

Further, let $\mathrm{Inn}_{\mathcal{R}}(\mathcal{A})$ be the ideal of $\mathrm{Der}_{\mathcal{F}}(\mathcal{A})$ of inner derivations determined by the elements of $\mathcal{R}$. Then the Lie $\mathcal{F}$-algebra $\mathrm{Der}_{\mathcal{F}}(\mathcal{A})$ is subdirectly irreducible with heart $\mathrm{Inn}_{\mathcal{R}}(\mathcal{A})$ (i.e., the intersection of all nonzero ideals of the Lie algebra $\mathrm{Der}_{\mathcal{F}}(\mathcal{A})$ is equal to $\mathrm{Inn}_{\mathcal{R}}(\mathcal{A})$). Finally, let $\mathcal{G}_{\mathcal{F}}(\mathcal{A})$ be the group of all (anti)automorphisms of the $\mathcal{F}$-algebra $\mathcal{A}$ and let

$$\mathrm{Aut}_{\mathcal{F}}(\mathrm{Der}_{\mathcal{F}}(\mathcal{A})), \quad \mathrm{Aut}_{\mathcal{F}}(\mathrm{Der}_{\mathcal{F}}(\mathcal{R})) \quad \text{and} \quad \mathrm{Aut}_{\mathcal{F}}(\mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}}))$$

be the groups of automorphisms of the Lie algebras $\mathrm{Der}_{\mathcal{F}}(\mathcal{A})$, $\mathrm{Der}_{\mathcal{F}}(\mathcal{R})$ and $\mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}})$ respectively. Then these four groups are canonically isomorphic.

In 1994 Kirkman, Procesi and Small [27] initiated the study of the Lie algebra V_q of inner derivations of the quantum Laurent polynomial algebra $\mathcal{A} = C_q[X, Y, X^{-1}, Y^{-1}]$ which they considered as a "q-analog" of the Virasoro algebra. They also described the group of automorphisms of the algebra $\mathcal{A}$ [27, Theorem 1.5]. The study of V_q was continued in [36] and [24], where it was shown that the Lie algebra of derivations of $\mathcal{A}$ is simple complete [24, Theorem 2] and is isomorphic to the Lie algebra of derivations of V_q [36, Theorem 1]. They also described the group $\mathrm{Aut}_{\mathcal{C}}(\mathrm{Der}_{\mathcal{C}}(\mathcal{A}))$ [24, Theorem 4]. We remark that both [24, Theorem 2] and [36, Theorem 1] are particular cases of Theorem 1.4 because $\mathcal{A}$ is a simple ring. Further, it is easy to see that $\mathcal{A}$ has no antiautomorphisms and so [24, Theorem 4] follows directly from both Theorem 1.4 and [27, Theorem 1.5].

2. Preliminary results

For the sake of completeness we now state a few results from [6, 7] upon which our paper is based. We first set in place some notation. In what follows $\mathcal{N}$ is the set of all nonnegative integers, $\mathcal{N}^* = \mathcal{N} \setminus \{0\}$, $\mathcal{F}$ is a commutative ring with unity, $\mathcal{Q}$ is an $\mathcal{F}$-algebra with 1 and with center $\mathcal{C}$, $\mathcal{R}$ is a nonempty subset of $\mathcal{Q}$, and $\mathcal{C}^*$ is the group of invertible elements of the ring $\mathcal{C}$. Given $n \in \mathcal{N}^*$ and $s_1, s_2, \ldots, s_n \in \mathcal{R}$, we set $\overline{s}_n = (s_1, s_2, \ldots, s_n) \in \mathcal{R}^n$ where $\mathcal{R}^n$ is the nth Cartesian power of $\mathcal{R}$. Further, let $m \in \mathcal{N}^*$, let $1 \leq i \leq m$ and let $G\colon \mathcal{R}^{m-1} \to \mathcal{Q}$ be a map (it is understood that G is a constant belonging to $\mathcal{Q}$ whenever $m = 1$). We define a map $G^i\colon \mathcal{R}^m \to \mathcal{Q}$ by the rule

$$G^i(\overline{x}_m) = G^i(x_1, x_2, \ldots, x_m) = G(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$$

for all $\overline{x}_m \in \mathcal{R}^m$. Now let $m \geq 2$, $1 \leq i < j \leq m$ and $H\colon \mathcal{R}^{m-2} \to \mathcal{Q}$. We define a map $H^{ij}\colon \mathcal{R}^m \to \mathcal{Q}$ by the rule

$$H^{ij}(\overline{x}_m) = H(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m)$$

for all $\overline{x}_m \in \mathcal{R}^m$ and set $H^{ji} = H^{ij}$.

Let $\mathcal{I}, \mathcal{J} \subseteq \{1, 2, \ldots, m\}$ and $E_i, F_j\colon \mathcal{R}^{m-1} \to \mathcal{Q}$, $i \in \mathcal{I}$, $j \in \mathcal{J}$. Consider **functional identities on** $\mathcal{R}$ of the following form:

$$(1) \qquad \sum_{i \in \mathcal{I}} E_i^i(\overline{x}_m) x_i + \sum_{j \in \mathcal{J}} x_j F_j^j(\overline{x}_m) = 0 \quad \text{for all } \overline{x}_m \in \mathcal{R}^m,$$

$$(2) \qquad \sum_{i \in \mathcal{I}} E_i^i(\overline{x}_m) x_i + \sum_{j \in \mathcal{J}} x_j F_j^j(\overline{x}_m) \in \mathcal{C} \quad \text{for all } \overline{x}_m \in \mathcal{R}^m.$$

Further, consider the following condition: there exist maps

$$(3) \qquad \begin{aligned} &p_{ij}\colon \mathcal{R}^{m-2} \to \mathcal{Q}, \quad i \in \mathcal{I},\ j \in \mathcal{J},\ i \neq j, \\ &\lambda_l\colon \mathcal{R}^{m-1} \to \mathcal{C}, \quad l \in \mathcal{I} \cup \mathcal{J}, \end{aligned}$$

such that

$$(4) \qquad \begin{aligned} E_i^i(\overline{x}_m) &= \sum_{\substack{j \in \mathcal{J}, \\ j \neq i}} x_j p_{ij}^{ij}(\overline{x}_m) + \lambda_i^i(\overline{x}_m), \\ F_j^j(\overline{x}_m) &= -\sum_{\substack{i \in \mathcal{I}, \\ i \neq j}} p_{ij}^{ij}(\overline{x}_m) x_i - \lambda_j^j(\overline{x}_m) \\ & i \in \mathcal{I}, \quad j \in \mathcal{J}, \\ & \lambda_l = 0 \quad \text{if } l \notin \mathcal{I} \cap \mathcal{J}. \end{aligned}$$

It is understood that all the p_{ij}'s are equal to 0 if $m = 1$.

Given $d \in \mathcal{N}^*$, following [6, Definition 1] the subset $\mathcal{R}$ of $\mathcal{Q}$ is called d**-free** if the following conditions are satisfied:

(i) For all $m \in \mathcal{N}^*$ and $\mathcal{I}, \mathcal{J} \subseteq \{1, 2, \ldots, m\}$ with $\max\{|\mathcal{I}|, |\mathcal{J}|\} \leq d$, we have that (1) implies (4).

(ii) For all $m \in \mathcal{N}^*$ and $\mathcal{I}, \mathcal{J} \subseteq \{1, 2, \ldots, m\}$ with $\max\{|\mathcal{I}|, |\mathcal{J}|\} \leq d - 1$, we have that (2) implies (4).

Assume for a while that $\mathcal{A}$ is a prime ring with noncentral Lie ideal $\mathcal{R}$, with maximal right ring of quotients $\mathcal{Q} = \mathcal{Q}_{mr}(\mathcal{A})$ and with Martindale centroid $\mathcal{C}$ (see [10, Chapter 2]). Suppose that $\mathcal{A}$ does not satisfy St_{2d}, the standard identity of degree $2d$. According to [6, Theorem 2.20]

(5) $\mathcal{R}$ is a d-free subset of $\mathcal{Q}$.

Let $\mathcal{R}$ be a d-free subset of $\mathcal{Q}$ with $d > 1$ and $q \in \mathcal{Q}$. It follows from [6, Remark 2.11(a)] that

(6) if $[q, \mathcal{R}] \subseteq \mathcal{C}$, then $q \in \mathcal{C}$.

THEOREM 2.1 ([6, Theorem 2.8]): *Let $d \in \mathcal{N}^*$ and let $\mathcal{B} \subseteq \mathcal{R} \subseteq \mathcal{Q}$ be subsets. Suppose that $\mathcal{B}$ is d-free. Then $\mathcal{R}$ is a d-free subset of $\mathcal{Q}$ as well.*

Let $\{x_1, x_2, \ldots, x_m\}$ be noncommuting variables, let $M = x_{i_1} x_{i_2} \cdots x_{i_k}$, $k \leq m$, be a multilinear monomial of degree k in the $\{x_{i_1}, \ldots, x_{i_k}\} \subseteq \{x_1, \ldots, x_m\}$ and let $\overline{s}_m \in \mathcal{R}^m$. We set

$$M(\overline{s}_m) = s_{i_1} s_{i_2} \cdots s_{i_k} \in \mathcal{Q}.$$

Next, let λ: $\mathcal{R}^{m-k} \to \mathcal{C}$ be a map. Then we denote by λM a map $\mathcal{R}^m \to \mathcal{Q}$ given by the rule

$$(\lambda M)(\overline{s}_m) = \lambda(s_{j_1}, s_{j_2}, \ldots, s_{j_{m-k}}) M(\overline{s}_m) \in \mathcal{Q},$$

where $\{j_1, j_2, \ldots, j_{m-k}\} = \{1, 2, \ldots, m\} \setminus \{i_1, i_2, \ldots, i_k\}$ and $j_1 < j_2 < \cdots < j_{m-k}$. The map λM is called **a multilinear quasi-monomial** of degree $\leq m$ with coefficient λ of the monomial M. If $\lambda \neq 0$, then we shall say that λM is a multilinear quasi-monomial of degree m. A sum $q(\overline{x}_m) = \sum \lambda_i M_i$ of multilinear quasi-monomials $\lambda_i M_i$ of degree $\leq m$ (where $M_i \neq M_j$ for $i \neq j$) is called **a multilinear quasi-polynomial** of degree $\leq m$ and each λ_i is called the coefficients of the monomial M_i in q. Finally, the coefficient of the monomial 1 in q is called **the constant term of** q.

The following two results are particular cases of [7, Theorems 1.1 and 2.11] respectively.

THEOREM 2.2: *Let $\mathcal{R}$ be a nonempty subset of $\mathcal{Q}$ and q a multilinear quasi-monomial of degree $\leq m$ such that $q(\overline{s}_m) = 0$ for all $\overline{s}_m \in \mathcal{R}^m$. Suppose that either the constant term of q is equal to 0 and $\mathcal{R}$ is an m-free subset of $\mathcal{Q}$, or $\mathcal{R}$ is an $(m+1)$-free subset of $\mathcal{Q}$. Then the coefficients of all monomials in q are equal to 0.*

THEOREM 2.3: *Let $\mathcal{R}$ be a submodule of the $\mathcal{F}$-module $\mathcal{Q}$ and let B: $\mathcal{R}^3 \to \mathcal{Q}$ be a trilinear map. Suppose that the following conditions are fulfilled:*

(1) *For all $\overline{s}_3 \in \mathcal{R}^3$*

$$B(s_1, s_2, s_3) + B(s_3, s_1, s_2) + B(s_2, s_3, s_1) \in \mathcal{C}.$$

(2) *For all $s_1, s_2, u, v \in \mathcal{R}$*

$$B(\overline{s}_2, [u, v]) - [u, B(\overline{s}_2, v)] - [B(\overline{s}_2, u), v] \in \mathcal{C}.$$

(3) *$\mathcal{R}$ is a 7-free subset of $\mathcal{Q}$.*

Then there exist a multilinear quasi-polynomial $q(s_1, s_2)$ without constant term and a homomorphism of $\mathcal{F}$-modules μ: $\mathcal{R}^3 \to \mathcal{C}$ such that $B(\overline{s}_3) = [q(\overline{s}_2), s_3] + \mu(\overline{s}_3)$ for all $\overline{s}_3 \in \mathcal{R}^3$. Finally, the coefficients of all the monomials in q are $\mathcal{F}$-multilinear.

Throughout the rest of the paper, $\overline{\mathcal{Q}} = \mathcal{Q}/\mathcal{C}$ is a factor Lie algebra of the Lie algebra $\mathcal{Q}$ by the Lie ideal $\mathcal{C}$, $\mathcal{R}$ is a Lie subalgebra of $\mathcal{Q}$ and $\mathcal{B} = \langle \mathcal{R} \rangle$ is the subalgebra of $\mathcal{Q}$ generated by $\mathcal{R}$. Given $x \in \mathcal{Q}$, we set $\overline{x} = x + \mathcal{C} \in \overline{\mathcal{Q}}$. We are now in a position to prove the following lemma which will play a crucial role in the paper.

LEMMA 2.4: *Let χ: $\mathcal{R} \to \overline{\mathcal{Q}}$ be an $\mathcal{F}$-linear map such that*

$$[x, y]^\chi = [x^\chi, \overline{y}] + [\overline{x}, y^\chi] \quad \text{for all } x, y \in \mathcal{R}.$$

Suppose that $\mathcal{R}$ is a 7-free subset of $\mathcal{Q}$ and $\mathcal{C}$ is a direct summand of the $\mathcal{C}$-module $\mathcal{Q}$. Next, assume that $\mathcal{R}$ is a Lie ideal of the $\mathcal{F}$-algebra $\mathcal{B}$ and $\frac{1}{2} \in \mathcal{F}$. Then there exists a derivation δ: $\mathcal{B} \to \mathcal{Q}$ of F-algebras such that $x^\chi = x^\delta + \mathcal{C} \in \overline{\mathcal{Q}}$ for all $x \in \mathcal{R}$. Further, if $\mathcal{R}^\chi \subseteq \overline{\mathcal{R}}$, then $\mathcal{B}^\delta \subseteq \mathcal{B}\mathcal{C} + \mathcal{C}$. Finally, if in addition $[\mathcal{R}, \mathcal{R}] = \mathcal{R}$, then $\mathcal{B}^\delta \subseteq \mathcal{B}$.

Proof: We note that

$$[x \circ y, z] + [y \circ z, x] + [z \circ x, y] = 0 \quad \text{and} \tag{7}$$

$$(x \circ y) \circ z - x \circ (y \circ z) = [y, [x, z]] \quad \text{for all } x, y, z \in \mathcal{Q}. \tag{8}$$

Our first step is to replace χ by an appropriate related map d: $\mathcal{R} \to \mathcal{Q}$ (thereby getting rid of cosets). Indeed, by assumption there exists a submodule $\mathcal{W}$ of the $\mathcal{C}$-module $\mathcal{Q}$ such that $\mathcal{Q} = \mathcal{W} \oplus \mathcal{C}$. Let π: $\mathcal{Q} \to \mathcal{W}$ be a canonical projection of

$\mathcal{C}$-modules. We define $d: \mathcal{R} \to \mathcal{W}$ as follows: $x^d = q$, where $\overline{q} = x^\chi$. One easily checks that d is a well-defined map of F-modules such that

$$[x,y]^d - [x^d, y] - [x, y^d] \in \mathcal{C} \quad \text{for all } x, y \in \mathcal{R}.$$

Our next step is to lift d to a map $\tau: \mathcal{B} \to \mathcal{W}$ such that

$$[x,y]^\tau - [x^\tau, y] - [x, y^\tau] \in \mathcal{C} \quad \text{for all } x, y \in \mathcal{B}.$$

Let Ω be the set of all pairs $(\mathcal{U}, \tau)$ such that:

(a) $\mathcal{U}$ is a Lie ideal of $\mathcal{B}$ containing $\mathcal{R}$;

(b) $\tau: \mathcal{U} \to \mathcal{W}$ is an $\mathcal{F}$-module map such that $[x,y]^\tau - [x^\tau, y] - [x, y^\tau] \in \mathcal{C}$ for all $x, y \in \mathcal{U}$;

(c) $x^\tau = x^d$ for all $x \in \mathcal{R}$.

Setting $(\mathcal{U}, \tau) \leq (\mathcal{V}, \sigma)$ if $\mathcal{U} \subseteq \mathcal{V}$ and $x^\tau = x^\sigma$ for all $x \in \mathcal{U}$, we partially order Ω. It follows from Zorn's lemma that the set Ω has a maximal element, say, $(\mathcal{U}, \tau)$. We claim that $\mathcal{U} = \mathcal{B}$. Since $\mathcal{U}$ is a Lie ideal of $\mathcal{B}$ containing $\mathcal{R}$ and $\frac{1}{2} \in \mathcal{F}$, it is enough to show that $\mathcal{U} \circ \mathcal{U} \subseteq \mathcal{U}$. To this end, we note that $\mathcal{U}$ is a 7-free subset of $\mathcal{Q}$ by Theorem 2.1 because $\mathcal{R} \subseteq \mathcal{U}$. Next we define a map $B: \mathcal{U}^3 \to \mathcal{Q}$ by the rule

$$B(x,y,z) = [x \circ y, z]^\tau - [x^\tau \circ y, z] - [x \circ y^\tau, z] - [x \circ y, z^\tau] \quad \text{for all } x, y, z \in \mathcal{U}.$$

Clearly B is a trilinear map and $B(x,y,z) = B(y,x,z)$ for all $x, y, z \in \mathcal{U}$. It follows from (7) that

$$B(x,y,z) + B(y,z,x) + B(z,x,y) = 0 \quad \text{for all } x, y, z \in \mathcal{U}. \tag{9}$$

Next, define a map $\epsilon: \mathcal{U}^2 \to \mathcal{C}$ by the rule $\epsilon(x,y) = [x,y]^\tau - [x^\tau, y] - [x, y^\tau]$ for all $x, y \in \mathcal{U}$. Then

$$[x,y]^\tau = [x^\tau, y] + [x, y^\tau] + \epsilon(x,y) \quad \text{for all } x, y \in \mathcal{U} \tag{10}$$

and so

$$\begin{aligned} B(x,y,[u,v]) =& [x \circ y, [u,v]]^\tau - [x^\tau \circ y, [u,v]] - [x \circ y^\tau, [u,v]] - [x \circ y, [u,v]^\tau] \\ =& \{[[x \circ y, u], v] + [u, [x \circ y, v]]\}^\tau - [[x^\tau \circ y, u], v] - [u, [x^\tau \circ y, v]] \\ & - [[x \circ y^\tau, u], v] - [u, [x \circ y^\tau, v]] - [x \circ y, [u^\tau, v] + [u, v^\tau]] \\ =& [[x \circ y, u]^\tau, v] + [[x \circ y, u], v^\tau] + [u^\tau, [x \circ y, v]] + [u, [x \circ y, v]^\tau] \\ & + \epsilon([x \circ y, u], v) + \epsilon(u, [x \circ y, v]) - [[x^\tau \circ y, u], v] - [u, [x^\tau \circ y, v]] \\ & - [[x \circ y^\tau, u], v] - [u, [x \circ y^\tau, v]] - [[x \circ y, u^\tau], v] - [u^\tau, [x \circ y, v]] \\ & - [[x \circ y, u], v^\tau] - [u, [x \circ y, v^\tau]] \\ =& [B(x,y,u), v] + [u, B(x,y,v)] + \epsilon([x \circ y, u], v) + \epsilon(u, [x \circ y, v]). \end{aligned}$$

Therefore

$$B(x,y,[u,v]) - [B(x,y,u),v] - [u,B(x,y,v)] \in \mathcal{C} \quad \text{for all } x,y,u,v \in \mathcal{U}$$

and whence (9) implies that all the assumptions of Theorem 2.3 are fulfilled. We conclude that there exist elements $a, b \in \mathcal{C}$, $\mathcal{F}$-linear maps $\mu, \nu: \mathcal{U} \to \mathcal{C}$ and a trilinear map $\omega: \mathcal{U}^3 \to \mathcal{C}$ such that

$$B(x,y,z) = [axy + byx + \mu(x)y + \nu(y)x, z] + \omega(x,y,z) \quad \text{for all } x,y,z \in \mathcal{U}.$$

Since $B(x,y,z) = B(y,x,z)$ for all $x,y,z \in \mathcal{U}$, Theorem 2.2 yields that $a = b$, $\mu = \nu$ and $\omega(x,y,z) = \omega(y,x,z)$ for all $x,y,z \in \mathcal{U}$. We now have

$$B(x,y,z) = [ax \circ y + \mu(x)y + \mu(y)x, z] + \omega(x,y,z) \quad \text{for all } x,y,z \in \mathcal{U}$$

and so

$$\text{(11)} \quad [x \circ y, z]^\tau = [ax \circ y + \mu(x)y + \mu(y)x + x^\tau \circ y + x \circ y^\tau, z] + [x \circ y, z^\tau] + \omega(x,y,z)$$

for all $x,y,z \in \mathcal{U}$.

We now set $\mathcal{V} = \mathcal{U} + \mathcal{U} \circ \mathcal{U}$. Clearly $\mathcal{R} \subseteq \mathcal{U} \subseteq \mathcal{V}$ and

$$[\mathcal{V}, \mathcal{B}] \subseteq [\mathcal{U}, \mathcal{B}] + \mathcal{U} \circ [\mathcal{U}, \mathcal{B}] \subseteq \mathcal{U} + \mathcal{U} \circ \mathcal{U} = \mathcal{V}$$

and so $\mathcal{V}$ is a Lie ideal of $\mathcal{B}$ containing both $\mathcal{R}$ and $\mathcal{U}$. Next, we define a map $\sigma: \mathcal{V} \to \mathcal{W}$ by the rule

$$\text{(12)} \quad \left\{ \sum_{i=1}^n x_i \circ y_i + z \right\}^\sigma = \left\{ \sum_{i=1}^n (ax_i \circ y_i + \mu(x_i)y_i + \mu(y_i)x_i + x_i^\tau \circ y_i + x_i \circ y_i^\tau) \right\}^\pi + z^\tau$$

for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n, z \in \mathcal{U}$. We claim that σ is a well-defined $\mathcal{F}$-linear map. Indeed, since $[x,y] = [x^\pi, y]$ for all $x, y \in \mathcal{Q}$, it follows from (11) that

$$\text{(13)} \quad \left[\sum_{i=1}^n x_i \circ y_i, t \right]^\tau = \left[\left(\sum_{i=1}^n x_i \circ y_i \right)^\sigma, t \right] + \left[\sum_{i=1}^n x_i \circ y_i, t^\tau \right] + \sum_{i=1}^n \omega(x_i, y_i, t)$$

for all $x_i, y_i, t \in \mathcal{U}$. Assume now that $\sum_{i=1}^n x_i \circ y_i + z = 0$ and take any $t \in \mathcal{R}$.

Then both (10) and (13) imply that

$$
\begin{aligned}
0 &= \left[\sum_{i=1}^{n} x_i \circ y_i + z, t\right]^{\tau} = \left[\left(\sum_{i=1}^{n} x_i \circ y_i\right)^{\sigma}, t\right] + \left[\sum_{i=1}^{n} x_i \circ y_i, t^{\tau}\right] \\
&\quad + \sum_{i=1}^{n} \omega(x_i, y_i, t) + [z^{\tau}, t] + [z, t^{\tau}] + \epsilon(z, t) \\
&= \left[\left(\sum_{i=1}^{n} x_i \circ y_i + z\right)^{\sigma}, t\right] + \left[\sum_{i=1}^{n} x_i \circ y_i + z, t^{\tau}\right] \\
&\quad + \sum_{i=1}^{n} \omega(x_i, y_i, t) + \epsilon(z, t) \\
&= \left[\left(\sum_{i=1}^{n} x_i \circ y_i + z\right)^{\sigma}, t\right] + \sum_{i=1}^{n} \omega(x_i, y_i, t) + \epsilon(z, t)
\end{aligned}
$$

and so $\left[(\sum_{i=1}^{n} x_i \circ y_i + z)^{\sigma}, t\right] \in \mathcal{C}$ for all $t \in \mathcal{R}$. It now follows from (6) that $(\sum_{i=1}^{n} x_i \circ y_i + z)^{\sigma} \in \mathcal{C}$. Since $\sigma\pi = \sigma$, we get $(\sum_{i=1}^{n} x_i \circ y_i + z)^{\sigma} = 0$ and so σ is well-defined. As τ, π and μ are $\mathcal{F}$-linear maps, σ is an $\mathcal{F}$-linear map as well.

Next, we claim that

$$
(14) \qquad [u, v]^{\sigma} - [u^{\sigma}, v] - [u, v^{\sigma}] \in \mathcal{C} \quad \text{for all } u, v \in \mathcal{V}.
$$

Indeed, it follows from (12) that

$$
(15) \qquad x^{\sigma} = x^{\tau} \quad \text{for all } x \in \mathcal{U}
$$

and so

$$
(16) \qquad [x, y]^{\sigma} - [x^{\sigma}, y] - [x, y^{\sigma}] = [x, y]^{\tau} - [x^{\tau}, y] - [x, y^{\tau}] \in \mathcal{C}
$$

for all $x, y \in \mathcal{U}$. Further, given $x, y, z \in \mathcal{U}$, we have $[x \circ y, z] \in \mathcal{U}$ and so both (15) and (13) yield that

$$
\begin{aligned}
&[x \circ y, z]^{\sigma} - [(x \circ y)^{\sigma}, z] - [x \circ y, z^{\sigma}] \\
(17) \qquad &= [x \circ y, z]^{\tau} - [(x \circ y)^{\sigma}, z] - [x \circ y, z^{\tau}] \in \mathcal{C}
\end{aligned}
$$

for all $x, y, z \in \mathcal{U}$. Finally, let $x, y, u, v \in \mathcal{U}$ and take $t \in \mathcal{R}$. Then

$$
[x \circ y, u \circ v] + [y \circ (u \circ v), x] + [(u \circ v) \circ x, y] = 0
$$

by (7) and so $[x \circ y, u \circ v] \in \mathcal{U}$. Clearly $[x \circ y, t], [u \circ v, t] \in \mathcal{U}$. Given $p, q \in \mathcal{Q}$, we shall write $p \equiv q$ provided that $p - q \in \mathcal{C}$. It now follows from both (16) and

(17) that

$$\begin{aligned}
[[x\circ y,u\circ v]^\sigma,t] &\equiv [[x\circ y,u\circ v],t]^\sigma - [[x\circ y,u\circ v],t^\sigma]\\
&=\{[[x\circ y,t],u\circ v]+[x\circ y,[u\circ v,t]]\}^\sigma - [[x\circ y,u\circ v],t^\sigma]\\
&\equiv[[x\circ y,t]^\sigma,u\circ v]+[[x\circ y,t],(u\circ v)^\sigma]+[(x\circ y)^\sigma,[u\circ v,t]]\\
&\quad+[x\circ y,[u\circ v,t]^\sigma]-[[x\circ y,u\circ v],t^\sigma]\\
&=[[(x\circ y)^\sigma,t],u\circ v]+[[x\circ y,t^\sigma],u\circ v]+[[x\circ y,t],(u\circ v)^\sigma]\\
&\quad+[(x\circ y)^\sigma,[u\circ v,t]]+[x\circ y,[(u\circ v)^\sigma,t]]+[x\circ y,[u\circ v,t^\sigma]]\\
&\quad-[[x\circ y,u\circ v],t^\sigma]\\
&=[[(x\circ y)^\sigma,u\circ v]+[x\circ y,(u\circ v)^\sigma],t]
\end{aligned}$$

and so

$$[[x\circ y,u\circ v]^\sigma-[(x\circ y)^\sigma,u\circ v]-[x\circ y,(u\circ v)^\sigma],t]\in\mathcal{C}\quad\text{for all } t\in\mathcal{R}.$$

It follows from (6) that

$$[x\circ y,u\circ v]^\sigma-[(x\circ y)^\sigma,u\circ v]-[x\circ y,(u\circ v)^\sigma]\in\mathcal{C}\quad\text{for all } x,y,u,v\in\mathcal{U}.$$

Since $\mathcal{V}=\mathcal{U}+\mathcal{U}\circ\mathcal{U}$, the above relation together with (16) and (17) imply that (14) is fulfilled. Therefore $(\mathcal{V},\sigma)\in\Omega$ and whence $\mathcal{V}=\mathcal{U}$ by the choice of $\mathcal{U}$. We see that $\mathcal{U}\circ\mathcal{U}\subseteq\mathcal{U}$ and so $\mathcal{U}=\mathcal{B}$. Since $\mathcal{V}=\mathcal{U}$, $\sigma=\tau$ by (15).

The final step is to find a derivation $\delta\colon\mathcal{B}\to\mathcal{Q}$ such that $x^\delta-x^\tau\in\mathcal{C}$ for all $x\in\mathcal{B}$ (thereby, in view of the first step, completing the proof of the first part of the lemma).

Now both (15) and (12) imply that

$$(x\circ y)^\tau=\{ax\circ y+\mu(x)y+\mu(y)x+x^\tau\circ y+x\circ y^\tau\}^\pi\quad\text{for all } x,y\in\mathcal{B}.$$

Setting $x^\delta=x^\tau+\frac{1}{2}\mu(x)$ and $\eta(x,y)=(x\circ y)^\delta-ax\circ y-x^\delta\circ y-x\circ y^\delta$ for all $x,y\in\mathcal{B}$, we conclude that δ is an $\mathcal{F}$-linear map and

$$(18)\qquad (x\circ y)^\delta=ax\circ y+x^\delta\circ y+x\circ y^\delta+\eta(x,y)\quad\text{for all } x,y\in\mathcal{B}.$$

Moreover, $\eta\colon\mathcal{B}^2\to\mathcal{C}$ because $\delta\pi=\tau$ and so $\eta(x,y)^\pi=0$. By (10) we have

$$\begin{aligned}
[x,y]^\delta &=[x,y]^\tau+\frac{1}{2}\mu([x,y])=[x^\tau,y]+[x,y^\tau]+\epsilon(x,y)+\frac{1}{2}\mu([x,y])\\
(19)\qquad &=[x^\delta,y]+[x,y^\delta]+\zeta(x,y)\quad\text{for all } x,y\in\mathcal{B},
\end{aligned}$$

where $\zeta(x,y) = \epsilon(x,y) + \frac{1}{2}\mu([x,y])$. Clearly $\zeta\colon \mathcal{B}^2 \to \mathcal{C}$. Making use of (18), we get

$$\begin{aligned}\{(x\circ y)\circ z\}^\delta =&a(x\circ y)\circ z + (x\circ y)^\delta \circ z + (x\circ y)\circ z^\delta + \eta(x\circ y, z)\\ =&2a(x\circ y)\circ z + (x^\delta\circ y)\circ z + (x\circ y^\delta)\circ z + 2\eta(x,y)z\\ &+ (x\circ y)\circ z^\delta + \eta(x\circ y,z) \quad \text{for all } x,y,z\in\mathcal{B}.\end{aligned} \tag{20}$$

Analogously,

$$\begin{aligned}\{x\circ(y\circ z)\}^\delta =&2ax\circ(y\circ z) + x^\delta\circ(y\circ z) + x\circ(y^\delta\circ z) + x\circ(y\circ z^\delta)\\ &+ 2\eta(y,z)x + \eta(x,y\circ z) \quad \text{for all } x,y,z\in\mathcal{B}.\end{aligned} \tag{21}$$

Subtracting (21) from (20) and making use of (8) we get

$$\begin{aligned}[y,[x,z]]^\delta =&2a[y,[x,z]] + [y^\delta,[x,z]] + [y,[x^\delta,z]] + [y,[x,z^\delta]]\\ &+ 2\eta(x,y)z - 2\eta(y,z)x + \eta(x\circ y,z) - \eta(x,y\circ z)\end{aligned} \tag{22}$$

for all $x,y,z\in\mathcal{B}$. Both (22) and (19) imply that

$$2a[y,[x,z]] + 2\eta(x,y)z - 2\eta(y,z)x + \eta(x\circ y,z) - \eta(x,y\circ z) - \zeta(y,[x,z]) = 0$$

for all $x,y,z\in\mathcal{B}$. Since $\mathcal{R}\subseteq\mathcal{B}$, Theorem 2.1 yields that $\mathcal{B}$ is a 7-free subset of $\mathcal{Q}$ and so Theorem 2.2 implies that, in particular, $a=0$ and $\eta=0$. Now (18) reads

$$(x\circ y)^\delta = x^\delta\circ y + x\circ y^\delta \quad \text{for all } x,y\in\mathcal{B}. \tag{23}$$

We now show that $\zeta=0$. To this end we note that both (19) and (23) imply that

$$\begin{aligned}(x\circ[x,y])^\delta =&x^\delta\circ[x,y] + x\circ[x,y]^\delta\\ =&x^\delta\circ[x,y] + x\circ[x^\delta,y] + x\circ[x,y^\delta] + 2\zeta(x,y)x\\ =&[(x^2)^\delta,y] + [x^2,y^\delta] + 2\zeta(x,y)x\\ =&[x^2,y]^\delta - \zeta(x^2,y) + 2\zeta(x,y)x\end{aligned}$$

for all $x,y\in\mathcal{B}$. Since $x\circ[x,y]=[x^2,y]$, we conclude that $2\zeta(x,y)x - \zeta(x^2,y) = 0$ for all $x,y\in\mathcal{B}$. Linearizing and making use of Theorem 2.2, we see that $\zeta=0$. That is to say $[x,y]^\delta = [x^\delta,y] + [x,y^\delta]$ for all $x,y\in\mathcal{B}$. It now follows from (23) that

$$\begin{aligned}(xy)^\delta =&\frac{1}{2}\{[x,y] + x\circ y\}^\delta = \frac{1}{2}\{[x^\delta,y] + [x,y^\delta] + x^\delta\circ y + x\circ y^\delta\}\\ =&x^\delta y + xy^\delta \quad \text{for all } x,y\in\mathcal{U}\end{aligned}$$

and so $\delta\colon \mathcal{B} \to \mathcal{Q}$ is a derivation of $\mathcal{F}$-algebras.

Now suppose that $\mathcal{R}^{\chi} \subseteq \overline{\mathcal{R}}$. Then, for $x \in \mathcal{R}$, $\overline{x^{\delta}} = x^{\chi} = \overline{r}$, $r \in \mathcal{R}$, and so $x^{\delta} \in \mathcal{R} + \mathcal{C}$ for $x \in \mathcal{R}$, whence $\mathcal{B}^{\delta} \subseteq \mathcal{BC} + \mathcal{C}$. Finally, if $\mathcal{R} = [\mathcal{R}, \mathcal{R}]$, then

$$\mathcal{R}^{\delta} = [\mathcal{R}, \mathcal{R}]^{\delta} \subseteq [\mathcal{R}^{\delta}, \mathcal{R}] \subseteq [\mathcal{R} + \mathcal{C}, \mathcal{R}] = [\mathcal{R}, \mathcal{R}] = \mathcal{R},$$

and so $\mathcal{B}^{\delta} \subseteq \mathcal{B}$. ∎

3. Proof of the main results

Before the proofs of Theorems 1.1 – 1.4 we make the following general observation. Let $\mathcal{Q} = \mathcal{Q}_{mr}(\mathcal{A})$. Since $\mathcal{A}$ is a prime algebra, $\mathcal{C}$ is a field and so it is a direct summand of the $\mathcal{C}$-module $\mathcal{Q}$. Further, by (5), $\mathcal{R}$ is a 7-free subset of $\mathcal{Q}$.

Proof: First we prove Theorem 1.1 and after that we continue with Theorem 1.2. Set $\overline{\mathcal{R}} = \mathcal{R}/\mathcal{Z}(\mathcal{R})$ and denote by π the canonical projection $\mathcal{R} \to \overline{\mathcal{R}}$. Setting $d = \delta\pi$, we see that $[x, y]^d = [x^d, \overline{y}] + [\overline{x}, y^d]$ for all $x, y \in \mathcal{R}$, where $\overline{x} = x^{\pi}$. It follows from Lemma 2.4 that there exists a derivation $D\colon \mathcal{B} \to \mathcal{BC} + \mathcal{C}$ of $\mathcal{F}$-algebras such that $x^d = x^D + \mathcal{C} \in \overline{\mathcal{Q}} = \mathcal{Q}/\mathcal{C}$ for all $x \in \mathcal{R}$. Set $\zeta(x) = x^D - x^{\delta}$ for all $x \in \mathcal{R}$. Then $\zeta(x)^{\pi} = (x^D + \mathcal{C}) - x^d = 0$ and so $\zeta\colon \mathcal{R} \to \mathcal{C}$. Clearly $x^D = x^{\delta} + \zeta(x)$ for all $x \in \mathcal{R}$. Next,

$$\begin{aligned}[x^{\delta}, y] + [x, y^{\delta}] =& [x^D, y] + [x, y^D] = [x, y]^D \\ =& [x, y]^{\delta} + \zeta([x, y]) = [x^{\delta}, y] + [x, y^{\delta}] + \zeta(x, y)\end{aligned}$$

and so $\zeta([x, y]) = 0$ for all $x, y \in \mathcal{R}$. The statement (c) of Theorem 1.1 follows from Lemma 2.4.

We now turn our attention to the proof of Theorem 1.2. We set $d = \pi\delta$ and note that $[x, y]^d = [x^d, \overline{y}] + [\overline{x}, y^d]$ for all $x, y \in \mathcal{R}$. The result now follows immediately from Lemma 2.4. ∎

We are now in a position to prove Theorems 1.3 and 1.4.

Proof: First we prove Theorem 1.3. We show that the map α is a well-defined isomorphism. Indeed, let $d \in \mathrm{Der}_{\mathcal{F}}(\mathcal{B})$. Since

$$\mathcal{R}^d = [\mathcal{R}, \mathcal{R}]^d \subseteq [\mathcal{R}, \mathcal{R}^d] \subseteq [\mathcal{R}, \mathcal{B}] \subseteq \mathcal{R},$$

we conclude that α is a well-defined homomorphism of Lie algebras. As $\mathcal{R}$ generates the $\mathcal{F}$-algebra $\mathcal{B}$, α is a monomorphism. The surjectivity of α follows from Theorem 1.1(a) and (b).

Next, we show that β is a well-defined isomorphism. To this end, let $d \in \mathrm{Der}_{\mathcal{F}}(\mathcal{R})$. Given $t \in \mathcal{Z}(\mathcal{R})$ and $x \in \mathcal{R}$, we have that

$$0 = [t, x]^d = [t^d, x] + [t, x^d] = [t^d, x] \quad \text{for all } x \in \mathcal{R}$$

and so $t^d \in \mathcal{Z}(\mathcal{R})$. Therefore $\mathcal{Z}(\mathcal{R})^d \subseteq \mathcal{Z}(\mathcal{R})$ whence d induces a Lie derivation on $\overline{\mathcal{R}}$. If the induced derivation is equal to 0, then $\mathcal{R}^d \subseteq \mathcal{Z}(\mathcal{R})$ and so $\mathcal{R}^d = [\mathcal{R}, \mathcal{R}]^d \subseteq [\mathcal{R}^d, \mathcal{R}] = 0$. Therefore β is a well-defined monomorphism of Lie algebras. Let $d' \in \mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}})$. By Theorem 1.2 there exists a derivation $D\colon \mathcal{B} \to \mathcal{B}$ of the $\mathcal{F}$-algebra $\mathcal{B}$ such that $(x + \mathcal{Z}(\mathcal{R}))^{d'} = x^D + \mathcal{Z}(\mathcal{R})$ in $\overline{\mathcal{R}}$ for all $x \in \mathcal{R}$. Clearly $D^\beta = d'$ whence β is an isomorphism of Lie algebras.

Let $d \in \mathrm{Der}_{\mathcal{F}}(\mathcal{B})$ with $[d, \mathrm{Inn}_{\mathcal{R}}(\mathcal{B})] = 0$. Then $[\mathrm{ad}_x, d] = 0$ for any inner derivation ad_x, $x \in \mathcal{R}$, and so

$$[x, y]^d = y^{\mathrm{ad}_x d} = y^{d\,\mathrm{ad}_x} = [x, y^d] \quad \text{for all } x \in \mathcal{R},\ y \in \mathcal{B},$$

forcing $[x^d, y] = 0$. Since $\mathcal{R}$ is 7-free, we conclude that $x^d \in \mathcal{C}$ whence $\mathcal{R}^d \subseteq \mathcal{C}$. Therefore $\mathcal{R}^d = [\mathcal{R}, \mathcal{R}]^d \subseteq [\mathcal{R}, \mathcal{R}^d] = 0$. Since $\mathcal{R}$ generates the algebra $\mathcal{B}$, we conclude that $d = 0$. Since the Lie algebras $\mathrm{Der}_{\mathcal{F}}(\mathcal{B})$ and $\mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}})$ are isomorphic and $\mathrm{Inn}_{\mathcal{R}}(\mathcal{B})$ is mapped onto the Lie ideal of inner derivations $\mathrm{Inn}_{\overline{\mathcal{R}}}(\overline{\mathcal{R}})$ of the Lie algebra $\overline{\mathcal{R}}$ under this isomorphism, we conclude that

$$[d, \mathrm{Inn}_{\overline{\mathcal{R}}}(\overline{\mathcal{R}})] \neq 0 \quad \text{for all } 0 \neq d \in \mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}}). \tag{24}$$

To complete the proof it is enough to show that the Lie algebra $\mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}})$ is complete. Let d be a Lie derivation of the Lie algebra $\mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}})$. Clearly the Lie algebras $\overline{\mathcal{R}}$ and $\mathrm{Inn}_{\overline{\mathcal{R}}}(\overline{\mathcal{R}})$ are isomorphic (via the map $\overline{x} \mapsto \mathrm{ad}_{\overline{x}}$). Since $[\overline{\mathcal{R}}, \overline{\mathcal{R}}] = \overline{\mathcal{R}}$, we see that $[\mathcal{I}, \mathcal{I}] = \mathcal{I}$ where $\mathcal{I} = \mathrm{Inn}_{\overline{\mathcal{R}}}(\overline{\mathcal{R}})$. It follows that $\mathcal{I}^d \subseteq \mathcal{I}$. First suppose that $\mathcal{I}^d = 0$. Given $D \in \mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}})$, we then have

$$0 = (\mathrm{ad}_{\overline{x}^D})^d = [\mathrm{ad}_{\overline{x}}, D]^d = [(\mathrm{ad}_{\overline{x}})^d, D] + [\mathrm{ad}_{\overline{x}}, D^d] \quad \text{for all } \overline{x} \in \overline{\mathcal{R}},$$

and so $[\mathrm{Inn}_{\overline{\mathcal{R}}}(\overline{R}), D^d] = 0$. By (24), $D^d = 0$ for all $D \in \mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}})$, that is to say, $d = 0$.

Finally, consider the general case. Since d induces a Lie derivation on $\mathrm{Inn}_{\overline{\mathcal{R}}}(\overline{\mathcal{R}})$, there exists $D \in \mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}})$ (from isomorphism $\overline{\mathcal{R}} \cong \mathrm{Inn}_{\overline{\mathcal{R}}}(\overline{\mathcal{R}})$) such that

$$(\mathrm{ad}_{\overline{x}})^d = \mathrm{ad}_{\overline{x}^D} = [\mathrm{ad}_{\overline{x}}, D] = (\mathrm{ad}_{\overline{x}})^{\mathrm{ad}_D}.$$

In other words, $\mathrm{Inn}_{\overline{\mathcal{R}}}(\overline{\mathcal{R}})^{d - \mathrm{ad}_D} = 0$. By the preceding argument $d - \mathrm{ad}_D = 0$ and so every derivation of $\mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}})$ is inner. By (24) the center of $\mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}})$ is equal to 0. The proof of Theorem 1.3 is thereby complete.

We now turn our attention to the proof of Theorem 1.4. Since $\mathcal{R} = [\mathcal{A}, \mathcal{A}]$, $[\mathcal{R}, \mathcal{R}] \not\subseteq \mathcal{Z}(\mathcal{A})$ because $\mathcal{A}$ does not satisfy St_{14}. It now follows from [16, Theorem 1.3] that $[\mathcal{R}, \mathcal{R}] = \mathcal{R}$. Finally, according to Corollary [16, p. 6], the subring of $\mathcal{A}$ generated by $\mathcal{R}$ is equal to $\mathcal{A}$. Since $\overline{\mathcal{R}} \cong \mathrm{Inn}_{\mathcal{R}}(\mathcal{A})$, [16, Theorem 1.12] implies that $\mathrm{Inn}_{\mathcal{R}}(\mathcal{A})$ is a simple Lie algebra. Given a nonzero ideal $\mathcal{L}$ of the Lie algebra $\mathrm{Der}_{\mathcal{F}}(\mathcal{A})$, by Theorem 1.3 we have $\mathcal{U} = [\mathcal{L}, \mathrm{Inn}_{\mathcal{R}}(\mathcal{A})] \neq 0$. Clearly $\mathcal{U}$ is a nonzero Lie ideal of the simple Lie algebra $\mathrm{Inn}_{\mathcal{R}}(\mathcal{A})$ and so $\mathcal{U} = \mathrm{Inn}_{\mathcal{R}}(\mathcal{A}) \subseteq \mathcal{L}$. That is to say, every nonzero ideal of the Lie algebra $\mathrm{Der}_{\mathcal{F}}(\mathcal{A})$ contains $\mathrm{Inn}_{\mathcal{R}}(\mathcal{A})$. Therefore $\mathrm{Der}_{\mathcal{F}}(\mathcal{A})$ is a subdirectly irreducible Lie algebra with heart $\mathrm{Inn}_{\mathcal{R}}(\mathcal{A})$. Since $\mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}}) \cong \mathrm{Der}_{\mathcal{F}}(\mathcal{A})$, we conclude that the Lie algebra $\mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}})$ is subdirectly irreducible and its heart is equal to $\mathrm{Inn}_{\overline{\mathcal{R}}}(\overline{\mathcal{R}})$.

By Theorem 1.3, $\mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}})$ is complete. We show that it is simple complete. Indeed, let $\mathcal{L}$ be a nonzero ideal of $\mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}})$ which is a complete Lie algebra. Then $\mathcal{L} \supseteq \mathrm{Inn}_{\overline{\mathcal{R}}}(\overline{\mathcal{R}})$ by the above result. Therefore any nonzero element of $\mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}})$ induces a nonzero derivation on $\mathcal{L}$ by (24). Since $\mathcal{L}$ is complete, we conclude that $\mathcal{L} = \mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}})$.

It has been proved that the Lie $\mathcal{F}$-algebras $\mathrm{Der}_{\mathcal{F}}(\mathcal{A})$, $\mathrm{Der}_{\mathcal{F}}(\mathcal{R})$ and $\mathrm{Der}_{\mathcal{F}}(\overline{\mathcal{R}})$ are canonically isomorphic. Therefore it is enough to show that the groups $\mathcal{G}_{\mathcal{F}}(\mathcal{A})$ and $\mathrm{Aut}_{\mathcal{F}}(\mathrm{Der}_{\mathcal{F}}(\mathcal{A}))$ are isomorphic under the map σ given by $d^{g^\sigma} = g^{-1}dg$ for all $d \in \mathrm{Der}_{\mathcal{F}}(\mathcal{A})$ and $g \in \mathcal{G}_{\mathcal{F}}(\mathcal{A})$. Clearly σ is a homomorphism of groups. If $g^\sigma = 1$, then $\mathrm{ad}_x = g^{-1}\mathrm{ad}_x g$ for all $x \in \mathcal{R}$. Suppose that g is an antiautomorphism. This leads to $x^g = -x + c$ for some $c \in \mathcal{C}$ and, since $\mathcal{R} = [\mathcal{R}, \mathcal{R}]$, we see that $x^g = -x$ for all $x \in \mathcal{R}$. Computing $[x^2, y]^g$, $y \in \mathcal{R}$, in two ways, we get the contradiction that x^2 is central. Therefore g must be an automorphism, whence it is easily seen that $g = 1$, and so σ is a monomorphism of groups.

To complete the proof, it is enough to show that σ is surjective. To this end, pick any automorphism τ of the Lie algebra $\mathrm{Der}_{\mathcal{F}}(\mathcal{A})$. Clearly it induces an automorphism of $\mathrm{Inn}_{\mathcal{R}}(\mathcal{A})$. We know that $\overline{\mathcal{R}} \cong \mathrm{Inn}_{\mathcal{R}}(\mathcal{A})$ via $\overline{x} \mapsto \mathrm{ad}_x$ and that $\mathcal{G}_{\mathcal{F}}(\mathcal{A}) \cong \mathrm{Aut}_{\mathcal{F}}(\overline{\mathcal{R}})$ via $g \mapsto g^\alpha$ where $\overline{x}^{g^\alpha} = \overline{x^g}$ if g is an automorphism and $\overline{x}^{g^\alpha} = -\overline{x^g}$ if g is an antiautomorphism (see [8, Theorem 1.4]). From this we see that $(\mathrm{ad}_x)^\tau = \mathrm{ad}_{\pm x^g} = g^{-1}\mathrm{ad}_x g = (\mathrm{ad}_x)^{g^\sigma}$, and so $\tau = g^\sigma$ on $\mathrm{Inn}_{\mathcal{R}}(\mathcal{A})$ for some $g \in \mathcal{G}_{\mathcal{F}}(\mathcal{A})$. Set $\chi = g^\sigma \tau^{-1}$. Obviously $(\mathrm{ad}_x)^\chi = \mathrm{ad}_x$ for all $x \in \mathcal{R}$. Given $d \in \mathrm{Der}_{\mathcal{F}}(\mathcal{A})$, we have that $[d, \mathrm{ad}_x] \in \mathrm{Inn}_{\mathcal{R}}(\mathcal{A})$ and so

$$[d, \mathrm{ad}_x] = [d, \mathrm{ad}_x]^\chi = [d^\chi, (\mathrm{ad}_x)^\chi] = [d^\chi, \mathrm{ad}_x] \quad \text{for all } x \in \mathcal{R}.$$

Therefore $[d^\chi - d, \mathrm{Inn}_{\mathcal{R}}(\mathcal{A})] = 0$ whence $d^\chi = d$ by Theorem 1.3. Thus $\chi = \mathrm{id}$, $\tau = g^\sigma$ and the proof is thereby complete. ∎

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References

[1] G. Ancochea, *Le théorème de von staundt en géometrie projective quaternionienne,* Journal für die reine und angewandte Mathematik **184** (1942), 192–198.

[2] G. Ancochea, *On semi-automorphisms of division algebras,* Annals of Mathematics **48** (1947), 147–153.

[3] K. I. Beidar, M. Brešar, M. A. Chebotar and W. S. Martindale, 3rd, *On Herstein's Lie map conjectures, I,* Transactions of the American Mathematical Society, to appear.

[4] K. I. Beidar, M. Brešar, M. A. Chebotar and W. S. Martindale, 3rd, *On Herstein's Lie map conjectures, II,* Journal of Algebra, to appear.

[5] K. I. Beidar and Y. Fong, *On additive isomorphisms of prime rings preserving polynomials,* Journal of Algebra **217** (1999), 650–667.

[6] K. I. Beidar and M. A. Chebotar, *On functional identities and d-free subsets of rings, I,* Communications in Algebra **28** (2000), 3925–3951.

[7] K. I. Beidar and M. A. Chebotar, *On functional identities and d-free subsets of rings, II,* Communications in Algebra **28** (2000), 3953–3972.

[8] K. I. Beidar and M. A. Chebotar, *On surjective Lie homomorphisms onto Lie ideals of prime rings,* Communications in Algebra, to appear

[9] K. I. Beidar, W. S. Martindale, 3rd and A. V. Mikhalev, *Lie isomorphisms in prime rings with involution,* Journal of Algebra **169** (1994), 304–327.

[10] K. I. Beidar, W. S. Martindale 3rd and A. V. Mikhalev, *Rings with Generalized Identities,* Marcel Dekker, New York, 1996.

[11] P. S. Blau, *Lie isomorphisms of non-GPI rings with involution,* Communications in Algebra **27** (1999), 2345–2373.

[12] M. Brešar, *Commuting traces of biadditive mappings, commutativity-preserving mappings and Lie mappings,* Transaction of the American Mathematical Society **335** (1993), 525–546.

[13] M. Brešar, W. S. Martindale 3rd and C. R. Miers, *Maps preserving n^{th} powers,* Communications in Algebra **26** (1998), 117–138.

[14] M. A. Chebotar, *On Lie automorphisms of simple rings of characteristic 2,* Fundamentalnaya i Prikladnaya Matematik **2** (1996), 1257–1268 (Russian).

[15] M. A. Chebotar, *On Lie isomorphisms in prime rings with involution,* Communications in Algebra **27** (1999), 2767–2777.

[16] I. N. Herstein, *Jordan homomorphisms*, Transaction of the American Mathematical Society **81** (1956), 331–351.

[17] I. N. Herstein, *Lie and Jordan Structure in simple, associative rings*, Bulletin of the American Mathematical Society **67** (1961), 517–531.

[18] R. A. Howland, *Lie isomorphisms of derived rings of simple rings*, Transaction of the American Mathematical Society **145** (1969), 383–396.

[19] L. K. Hua, *On the automorphisms of a sfield*, Proceeding of the National Academy of Sciences of the United States of America **35** (1949), 386–389.

[20] L. K. Hua, *A theorem on matrices over an sfield and its applications*, Journal of the Chinese Mathematical Society (N.S.) **1** (1951), 110–163.

[21] D. R. Jacobs, *Lie derivations on skew elements of simple rings with involution*, Ph.D dissertation, University of Massachusetts, 1973.

[22] N. Jacobson, *Lie Algebras*, Dover Publications, New York, 1979.

[23] N. Jacobson and C. Rickart, *Jordan homomorphisms of rings*, Transaction of the American Mathematical Society **69** (1950), 479–502.

[24] C. Jiang and D. Meng, *The derivation algebra of the associative algebra* $C_q[X,Y,X^{-1},Y^1]$, Communications in Algebra **26** (1998), 1723–1736.

[25] I. Kaplansky, *Semi-automorphisms of rings*, Duke Mathematical Journal **14** (1947), 521–527.

[26] E. Killiam, *Lie derivations on skew elements in prime rings with involution*, Canadian Mathematical Bulletin **30** (1987), 344–350.

[27] E. Kirkman, C. Procesi and L. Small, *A q-analog for Virasoro algebra*, Communications in Algebra **22** (1994), 3755–3774.

[28] L. A. Lagutina, *On Lie automorphisms of simple associative algebras with involution*, Russian Mathematical Surveys **47** (1992), 222–223 (English. Russian original).

[29] W. S. Martindale, 3rd, *Lie isomorphisms of primitive rings*, Proceedings of the American Mathematical Society **14** (1963), 909–916.

[30] W. S. Martindale, 3rd, *Lie derivations of primitive rings*, Michigan Journal of Mathematics **11** (1964), 183–187.

[31] W. S. Martindale, 3rd, *Lie isomorphisms of simple rings*, Journal of the London Mathematical Society **44** (1969), 213–221.

[32] W. S. Martindale, 3rd, *Lie isomorphisms of prime rings*, Transaction of the American Mathematical Society **142** (1969), 437–455.

[33] W. S. Martindale, 3rd, *Lie isomorphisms of the skew elements of a simple ring with involution*, Journal of Algebra **36** (1975), 408–415.

[34] W. S. Martindale, 3rd, *Lie isomorphisms of the skew elements of a prime ring with involution,* Communications in Algebra **4** (1976), 927–977.

[35] W. S. Martindale, 3rd, *Lie and Jordan mappings,* Contemporary Mathematics **13** (1982), 173–177.

[36] D. Meng and C. Jiang, *The derivation algebra and the universal central extensions of the q-analog of the Virasoro-like algebra,* Communications in Algebra **26** (1998), 1335–1346.

[37] C. R. Miers, *Lie triple derivations of von Neumann algebras,* Proceedings of the American Mathematical Society **71** (1978), 57–61.

[38] M. P. Rosen, *Lie isomorphisms of a certain class of prime rings,* Journal of Algebra **89** (1984), 291–317.

[39] M. F. Smiley, *Jordan homomorphisms onto prime rings,* Transaction of the American Mathematical Society **84** (1957), 426–429.

[40] G. A. Swain, *Lie derivations of the skew elements of prime rings with involution,* Journal of Algebra **184** (1996), 679–704.

[41] G. A. Swain and P. S. Blau, *Lie derivations in prime rings with involution,* Canadian Mathematical Bulletin **42** (1999), 401–411.